Method of consecutive iterations for solving nonlinear algebraic equations

The equation $f(x)=0$ is transformed into the equivalent one: $x=\varphi(x)$. If the function $\varphi(x)$ is such that for every $x\in[a,b]$ the inequality $|\varphi'(x)|\le q<1$ is fulfilled then the iteration process

$$x_{n+1}=\varphi(x_n)\,,\ n=0,1,2,\ldots$$

is convergent towards the root of the equation $f(x)=0$ for every initial $x_0\in(a,b)$ and the following evaluation is also valid

$$\left|x^*-x_n\right|\le\frac{q^n}{1-q}|x_1-x_0|.$$

If f' has a constant sign (for example $f'>0$ otherwise we consider the equation $-f(x)=0$) and $0<m_1\le f'(x)\le M_1$ for $\forall x\in[a,b]$, then the function $\varphi(x)=x-f'(x)/M_1$ guarantees convergence at the rate of a geometric progression with quotient $q=1-\dfrac{m_1}{M_1}<1$.

Example. Construct a convergent iteration process using the method of consecutive approximations (MCA) to find a real root for the equation $x^3+x-12=0$. Find x_3 (third iteration) using the constructed method, assuming as initial approximation the middle of the chosen interval and evaluate the error. How many iterations would guarantee an accuracy of 10^{-6} for the same initial approximation?

Solution:

Obviously the root of the equation is a positive number (why?). We enter the function $f(x)=x^3+x-12$ into a table for positive whole numbers.

x	1	2	3
$f(x)$	-10	-2	18

This way we localize the root in the interval [2; 3]. In order to get a convergent iteration process we solve the equation with regard to x from x^3:

$x = (12 - x)^{1/3}$; $\varphi(x) = (12 - x)^{1/3}$. In order to prove the convergence we calculate:

$$\varphi'(x) = \frac{1}{3}(12 - x)^{-2/3}.(-1) = \frac{-1}{3\sqrt[3]{(12 - x)^2}}$$

$$\max_{[2,3]}|\varphi'(x)| = \max_{[2,3]}\left|\frac{-1}{3\sqrt[3]{(12 - x)^2}}\right| = \frac{1}{3\sqrt[3]{9^2}} = 0{,}077 \approx 0{,}08,$$

i.e. the iteration process $x_{n+1} = \sqrt[3]{12 - x_n}$ would be convergent at the rate of a geometric progression with quotient $q = 0{,}08$ for every choice of the initial guess $x_0 \in [2{,}3]$.

In order to evaluate the error of the third approximation we first calculate x_1 when $x_0 = 2{,}5$, $x_1 = (12 - 2{,}5)^{1/3} = 2{,}117912$.

Then $\left|x^* - x_3\right| \le \frac{0{,}08^3}{1 - 0{,}08}|2{,}117912 - 2{,}5| = 0{,}00021...$; or the error of the third approximation would be around 0,0002 and is sufficient to carry out the intermediate calculations up to the fourth digit after the decimal comma. The calculations are given in the following table:

n	x_n	$\varphi(x_n)$
0	2,5	2,1179
1	2,1179	2,1459
2	2,1459	2,1439
3	2,1439	

Answer: $x^* \approx x_3 = 2{,}1439 \approx 2{,}144$.

We can note that in this case the consecutive approximations oscillate around the root of the equation (their sequence is not monotonous) and for the evaluation of the error we can use the “property of the middle”. This is due to the fact that $\varphi'(x) < 0$ for $\forall x \in [2{,}3]$.

In order to evaluate the number of iterations needed to guarantee an accuracy of 10^{-6}, we solve in regard to n the following inequality:

$$\frac{0{,}08^n}{1-0{,}08}\left|2{,}117912-2{,}5\right| \le 10^{-6} \quad \text{or} \quad 0{,}08^n \le \frac{0{,}92.10^{-6}}{0{,}382088},$$

from where $n \ge \log\frac{0{,}92.10^{-6}}{0{,}382088} / \log 0{,}08 = 5{,}122$, i.e. by the sixth iteration we would have a guaranteed accuracy of 10^{-6}.

Note. When solving the inequality above, we took its logarithm without specifying the base, i.e. we can use ln, lg, $\log_2$ depending on the available possibilities. The inequality changes its direction after its division by log 0,08 because log0,08<0 for a random base >1.

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